Passage 24 (A)

TAKASU Fugo
Department of Environmental Science NWU

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1 Logistic model

Let N_t represent the population size in year t. If the population increases by a constant factor, say r every year, the population size in the next year $t+1$ is given as

$$N_{t+1} = rN_t \tag{1}$$

Solution of this difference equation is

$$N_t = N_0 r^t \tag{2}$$

where N_0 is initial population size in year $t = 0$. If $r > 1$, the population exponentially increases. Otherwise, it decreases exponentially.

In human demography, r is never kept a constant. It can change with time as we have seen for many countries. Many factors can influence the change of r.

As a variant of the exponential model, we assume that r depends on the population size. Many animals including human being depend on "resources" such as food, water and living space. So it would make send to assume that r depends on N_t. If r linearly decreases with N_t, we assume that

$$r(N_t) = r\left(1 - \frac{N_t}{K}\right)$$

where K is a constant. Then we have the following difference equation

$$N_{t+1} = r\left(1 - \frac{N_t}{K}\right) N_t \tag{3}$$

This difference equation is called "Logistic model". Once we give N_0, we obtain a unique sequence $\{N_0, N_1, N_2, \cdots\}$. However, we cannot derive the solution in explicit form.

An approach to study this kind of "non-linear difference equation" is to focus on "equilibrium" that does not change with time. Equilibrium N^* satisfies $N_{t+1} = N_t = N^*$. Thus from

$$N^* = r\left(1 - \frac{N^*}{K}\right) N^* \tag{4}$$

we have equilibrium explicitly as

$$N^* = 0, \quad K\frac{r-1}{r} \tag{5}$$

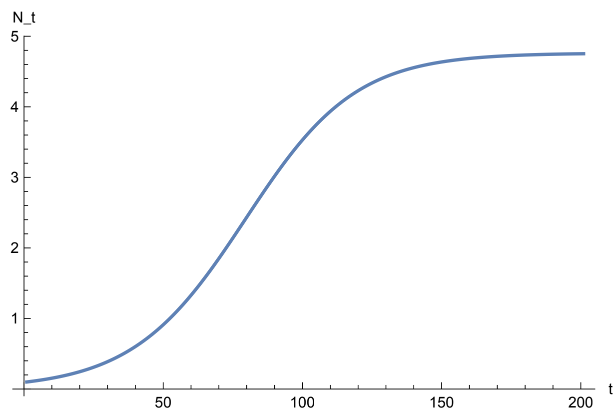

Figure 1: A typical dynamics of the logistic model with $N_0 = 0.1$. $r = 1.05, K = 100, N^* = K(r-1)/r = 4.7619$.

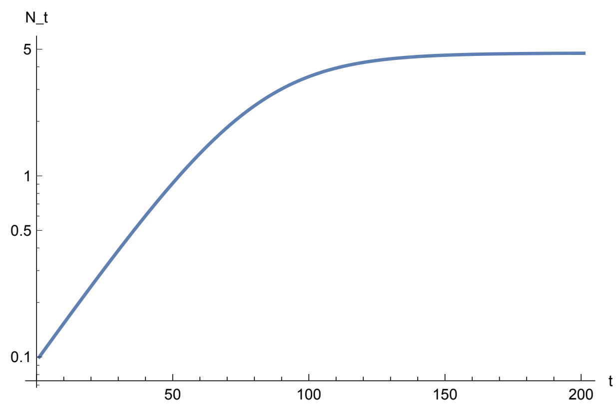

Figure 2: The same above but in logarithmic scale.

It can be shown that any $N_0 > 0$ converges to $N^* = K(r-1)/r$ when r is slightly larger than 1.

In the logistic model, N_t increases nearly exponentially when the population size is small (resource is abundant and it increases exponentially). As N_t becomes larger, however, increase slows down and eventually converges to N^*. The graph looks like S shaped.

The logistic model is often used to represent population dynamics in a limited resource.