

Passage 24 A #3

How does a population grow in size? - A mathematical approach

■ Geometric growth

Let N_t represent the population size at time t . If the population increases by 1% annually, the size at time $t+1$, N_{t+1} , is given by

$$N_{t+1} = N_t + 0.01 N_t = 1.01 N_t \quad (1)$$

This is a difference equation that relates N_t to N_{t+1} . We know that the solution is given by

$$N_t = N_0 1.01^t \quad (2)$$

using the initial population size at $t = 0$, N_0 . This is a “geometric” sequence with the common ratio 1.01.

In general, if the population increases by $a\%$ annually, the size N_{t+1} follows

$$N_{t+1} = N_t + a/100 N_t = (1 + a/100) N_t \quad (3)$$

and the solution is

$$N_t = N_0 r^t \quad (4)$$

with the common ratio is r . We know that if $a > 0$ and $r > 1$, N_t exponentially increases. If $a < 0$ and $0 < r < 1$, it exponentially decreases to zero.

Assume a population where each individual gives birth to a certain number of offspring and survives a unit time (one year) with a certain survival probability. We ignore sex (male and female). Just assume a population of such as bacteria that proliferate by dividing themselves into several new individuals per unit time.

Then the population size at time $t+1$, N_{t+1} , is given by

$$N_{t+1} = b N_t + s N_t \quad (5)$$

where the first term in the right hand side represents the number of new offspring recruited into the population in the next time step and the second term represents the number of individuals who survive one unit time.

b is the number of offspring that survives to the next year per individual and s is the survival probability per unit time. This difference equation can be written as

$$N_{t+1} = r N_t \quad (6)$$

as $r = b + s$.

The above equation (5) can be rearranged as

$$\Delta N = N_{t+1} - N_t = b N_t + s N_t - N_t = b N_t - (1 - s) N_t \quad (7)$$

Thus, the net change in the population size N between successive times, ΔN , is given by the increase by birth minus the decrease by death. Note that $1 - s$ represents the probability of death.

ΔN being positive means that the population increases, and it being negative means it decreases. In other words, $b + s - 1 > 0$ means increase and $b + s - 1 < 0$ means decrease; the same condition as $r = b + s > 1$ or $r < 1$.

Now let's see how a geometric sequence looks like.

Example 0 : Exponential model

Let's set r to be 1.05, i.e., the population increases by 5% every year (Annual growth rate is 5%).

```
In[1]:= fn[x_] := r x /. r -> 1.05
```

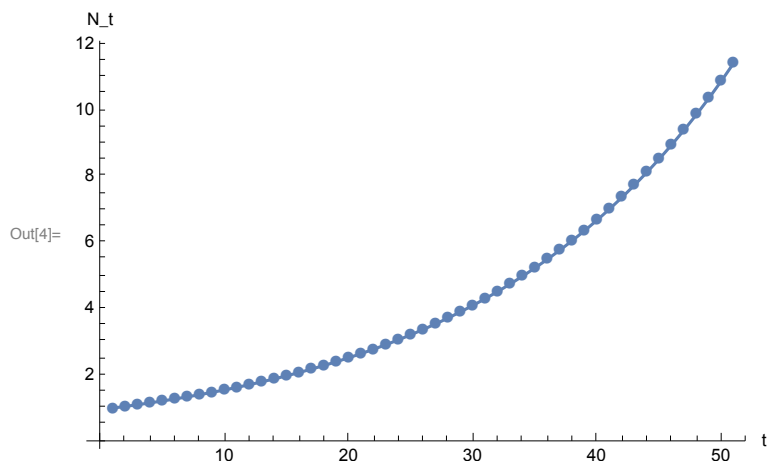
```
In[2]:= fn[1]
```

```
Out[2]:= 1.05
```

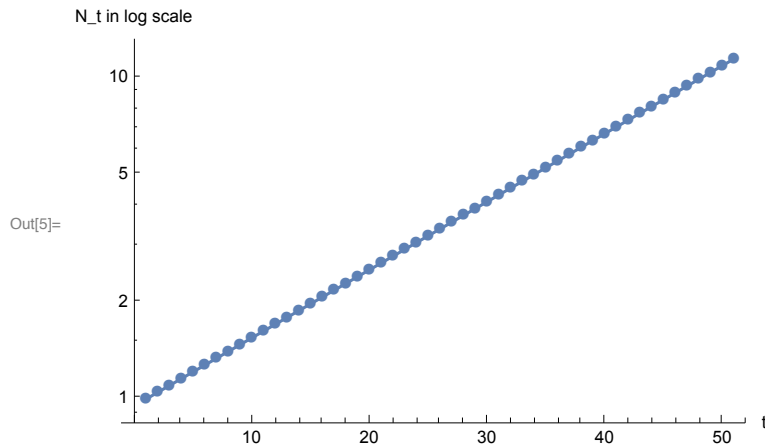
```
In[3]:= data = NestList[fn, 1, 50]
```

```
Out[3]:= {1, 1.05, 1.1025, 1.15763, 1.21551, 1.27628, 1.3401, 1.4071, 1.47746, 1.55133,
1.62889, 1.71034, 1.79586, 1.88565, 1.97993, 2.07893, 2.18287, 2.29202,
2.40662, 2.52695, 2.6533, 2.78596, 2.92526, 3.07152, 3.2251, 3.38635, 3.55567,
3.73346, 3.92013, 4.11614, 4.32194, 4.53804, 4.76494, 5.00319, 5.25335,
5.51602, 5.79182, 6.08141, 6.38548, 6.70475, 7.03999, 7.39199, 7.76159,
8.14967, 8.55715, 8.98501, 9.43426, 9.90597, 10.4013, 10.9213, 11.4674}
```

```
In[4]:= ListPlot[data, Joined -> True, PlotMarkers -> Automatic, AxesLabel -> {"t", "N_t"}]
```



```
In[5]:= ListLogPlot[data, Joined → True,
  PlotMarkers → Automatic, AxesLabel → {"t", "N_t in log scale"}]
```



IMPORTANT: Geometric population growth is translated to a straight line when plotted in the logarithmic scale against time t , or a linear function of t .

This can be shown as follows. Take logarithm of both the sides of eq. (3)

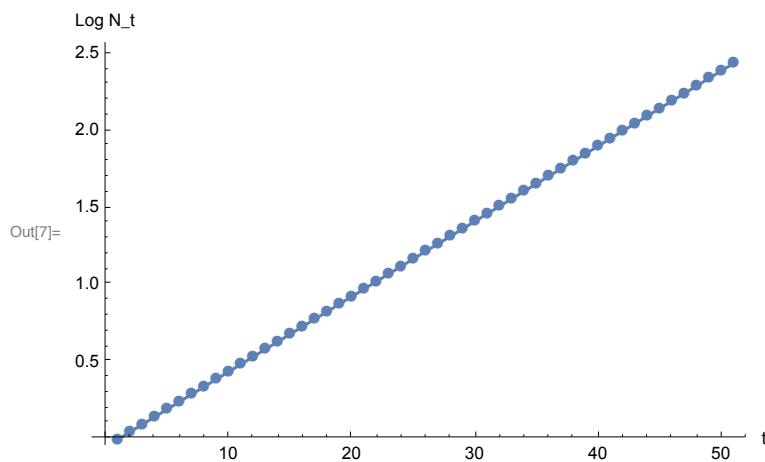
$$\log N_t = \log N_0 r^t = \log N_0 + \log r^t = \log N_0 + t \log r$$

Therefore, $\log N_t$ is a linear function of time t , i.e., a straight line with the slope $\log r$ when plotted against time t .

```
In[6]:= dataLog = Log[data]
```

```
Out[6]= {0, 0.0487902, 0.0975803, 0.14637, 0.195161, 0.243951, 0.292741, 0.341531,
  0.390321, 0.439111, 0.487902, 0.536692, 0.585482, 0.634272, 0.683062, 0.731852,
  0.780643, 0.829433, 0.878223, 0.927013, 0.975803, 1.02459, 1.07338, 1.12217,
  1.17096, 1.21975, 1.26854, 1.31733, 1.36612, 1.41491, 1.4637, 1.5125, 1.56129,
  1.61008, 1.65887, 1.70766, 1.75645, 1.80524, 1.85403, 1.90282, 1.95161, 2.0004,
  2.04919, 2.09798, 2.14677, 2.19556, 2.24435, 2.29314, 2.34193, 2.39072, 2.43951}
```

```
In[7]:= ListPlot[dataLog, PlotMarkers → Automatic,
  Joined → True, AxesLabel → {"t", "Log N_t"}]
```



If we know the slope of the straight line in the logarithmic scale, we can obtain the common ratio r from the slope.

```
In[8]:= Fit[dataLog, {1, t}, t]
```

```
Out[8]= -0.0487902 + 0.0487902 t
```

```
In[9]:= Exp[0.0487902]
```

```
Out[9]= 1.05
```

```
In[10]:= Log[1.05]
```

```
Out[10]= 0.0487902
```

Example 1: Collared doves

The doves brought in England as a pet increased from 4 to 18,855 in 10 years.

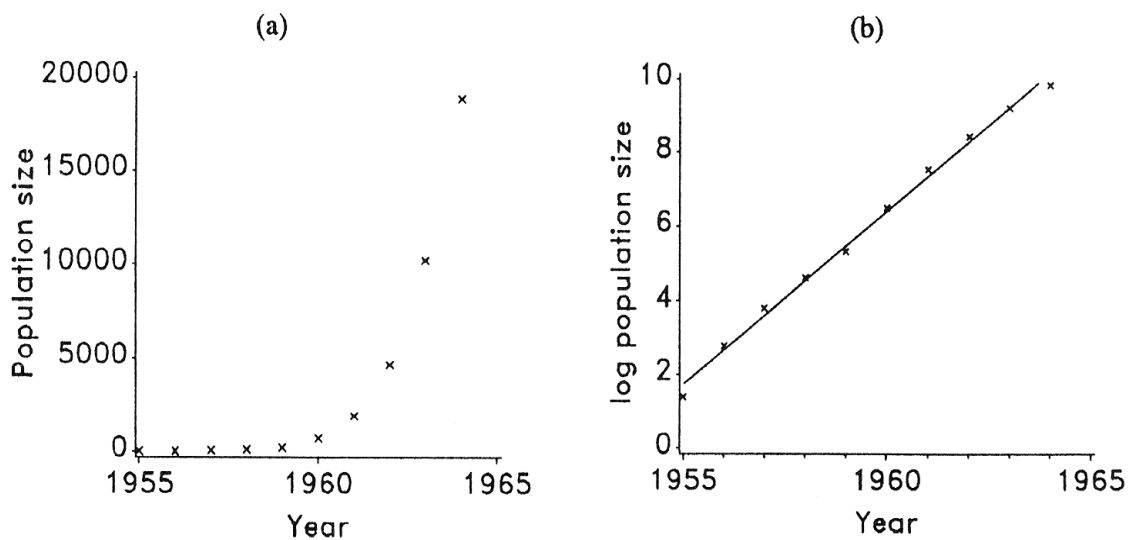


Figure 1.6 Exponential growth in a population of collared doves. (a) Increase in the number of adults plus young at end of breeding season; (b) approximately linear increase of log numbers with fitted straight line for exponential growth and geometric rate of increase 2.66 per annum.

Log of the population size almost linearly increases with the slope 0.978. This means that the doves increased nearly exponentially with the common ratio $e^{0.978} = 2.66$ every year.

Example 2 : A pheasant population in an island

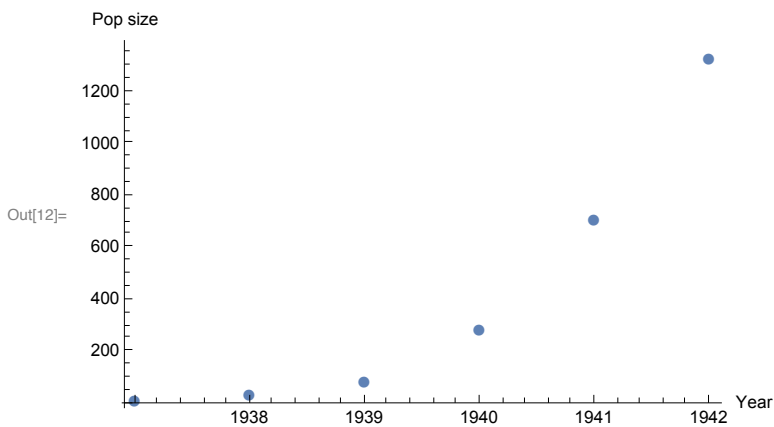
Lack (1954) observed the number of pheasants in an island as follows.

```
In[11]:= data =
```

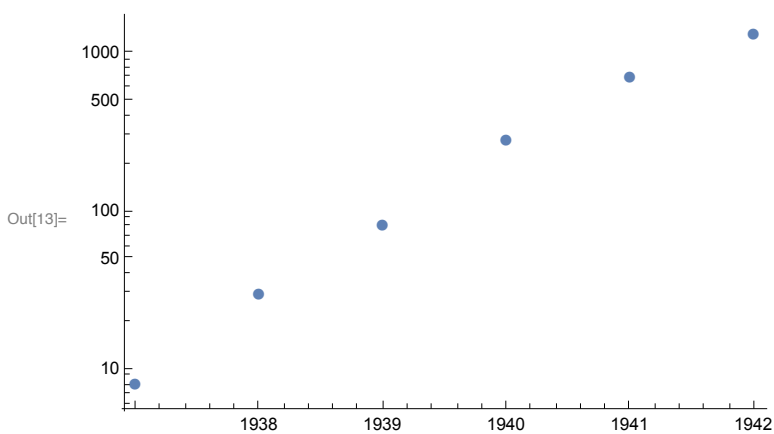
```
{ {1937, 1938, 1939, 1940, 1941, 1942}, {8, 30, 81, 282, 705, 1325} } // Transpose
```

```
Out[11]= { {1937, 8}, {1938, 30}, {1939, 81}, {1940, 282}, {1941, 705}, {1942, 1325} }
```

In[12]:= **ListPlot**[data, PlotMarkers → Automatic, AxesLabel → {"Year", "Pop size"}]



In[13]:= **ListLogPlot**[data, PlotMarkers → Automatic]



In logarithmic scale, it almost linearly increases with time t. Let's examine the slope of the line.

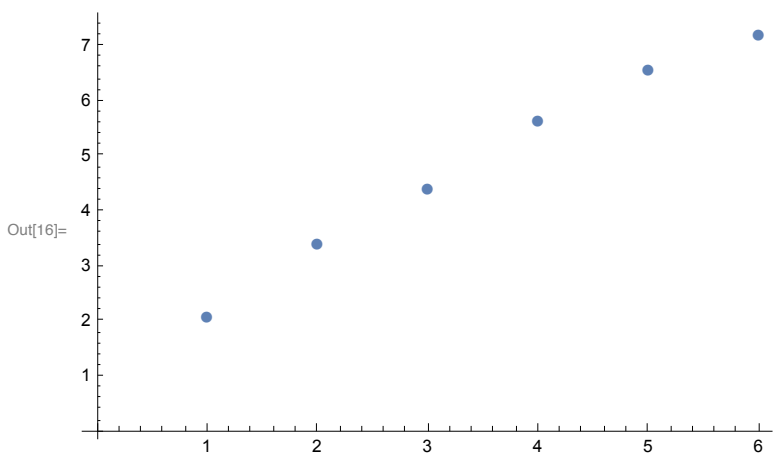
In[14]:= **dataPopSize** = {8, 30, 81, 282, 705, 1325}

Out[14]= {8, 30, 81, 282, 705, 1325}

In[15]:= **dataPopSizeLog** = **Log**[dataPopSize]

Out[15]= {Log[8], Log[30], Log[81], Log[282], Log[705], Log[1325]}

In[16]:= **gLog** = **ListPlot**[dataPopSizeLog, PlotMarkers → Automatic]

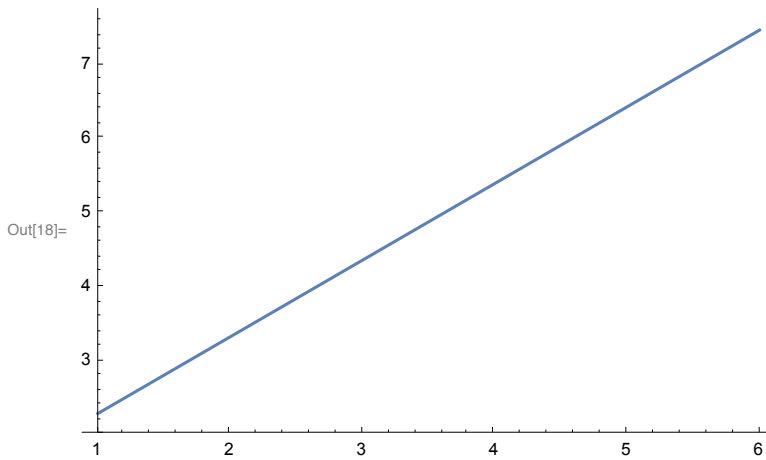


We can obtain the slope by "regression analysis", a method of statistics.

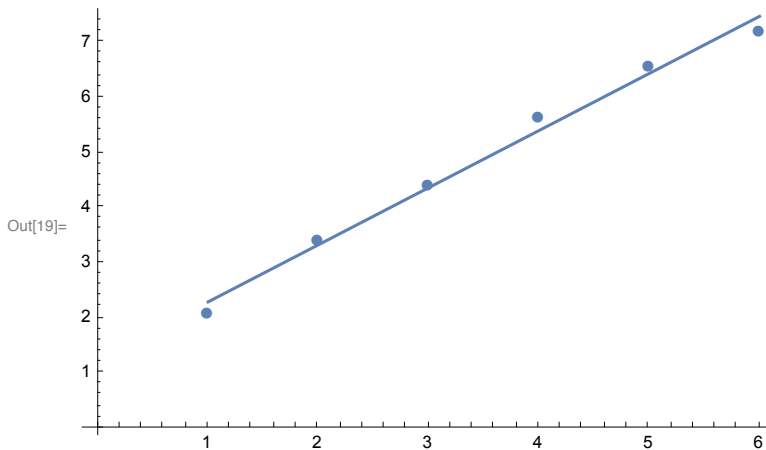
```
In[17]:= reg = Fit[dataPopSizeLog, {1, t}, t]
```

```
Out[17]= 1.25068 + 1.0362 t
```

```
In[18]:= gReg = Plot[reg, {t, 1, 6}]
```



```
In[19]:= Show[gLog, gReg]
```



```
In[20]:= Coefficient[reg, t]
```

```
Out[20]= 1.0362
```

```
In[21]:= Exp[Coefficient[reg, t]]
```

```
Out[21]= 2.81849
```

The pheasant in the island increased roughly exponentially with the common ratio 2.82.

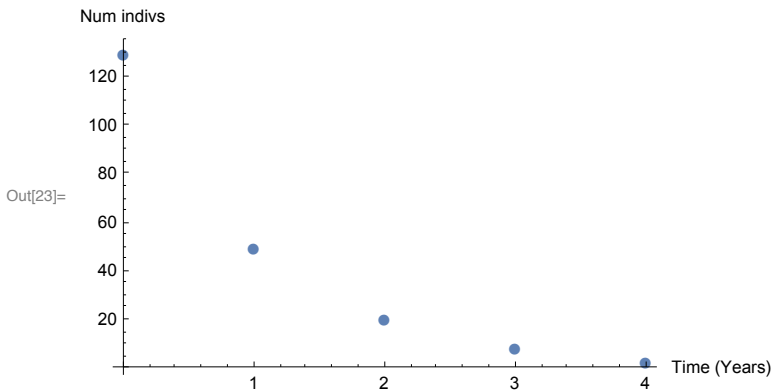
Example 3 : Survival of robins

Lack (1965) observed how robins survived for several years and the result is shown as follows.

```
In[22]:= dataNumAlive = Transpose[{{0, 1, 2, 3, 4}, {129, 49, 20, 8, 2}}]
```

```
Out[22]= {{0, 129}, {1, 49}, {2, 20}, {3, 8}, {4, 2}}
```

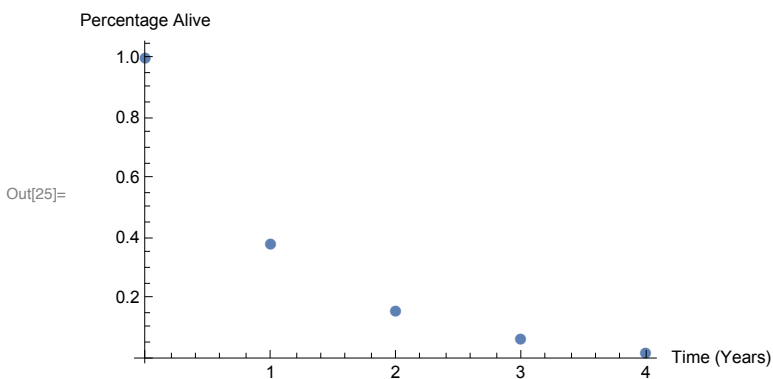
```
In[23]:= ListPlot[dataNumAlive, PlotMarkers → Automatic,
  AxesLabel → {"Time (Years)", "Num indivs"}]
```



```
In[24]:= dataPercentageAlive = Transpose[{{0, 1, 2, 3, 4}, {129, 49, 20, 8, 2} / 129}]
```

```
Out[24]= {{0, 1}, {1, 49/129}, {2, 20/129}, {3, 8/129}, {4, 2/129}}
```

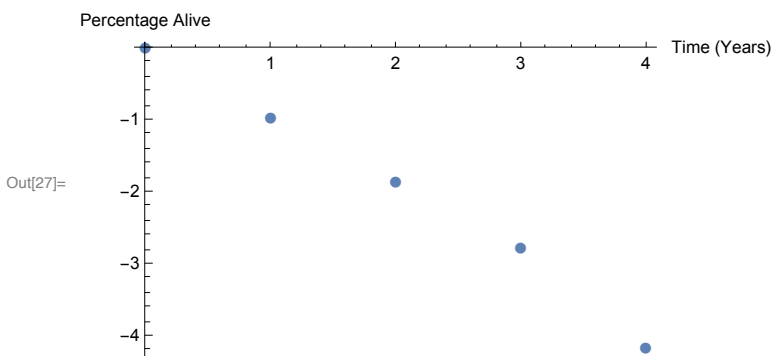
```
In[25]:= ListPlot[dataPercentageAlive, PlotMarkers → Automatic,
  AxesLabel → {"Time (Years)", "Percentage Alive"}]
```



```
In[26]:= dataPercentageAliveLog = Transpose[{{0, 1, 2, 3, 4}, Log[{129, 49, 20, 8, 2} / 129]}]
```

```
Out[26]= {{0, 0}, {1, -Log[129/49]}, {2, -Log[129/20]}, {3, -Log[129/8]}, {4, -Log[129/2]}}
```

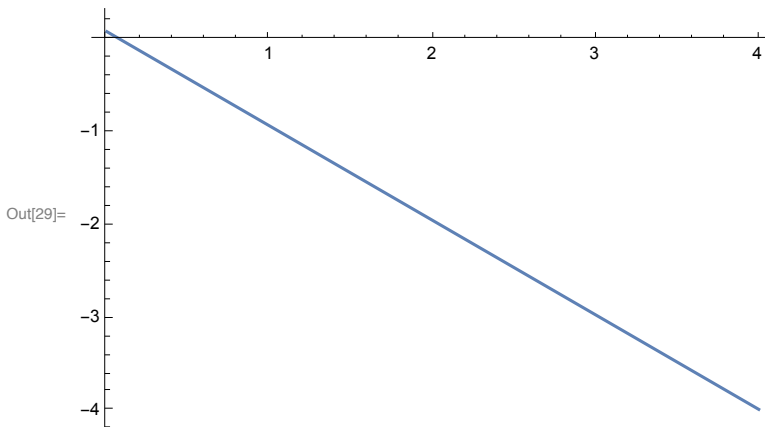
```
In[27]:= g1 = ListPlot[dataPercentageAliveLog, PlotMarkers → Automatic,
  AxesLabel → {"Time (Years)", "Percentage Alive"}]
```



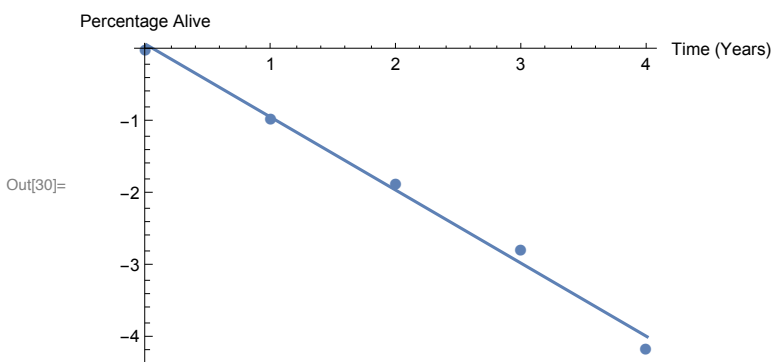
```
In[28]:= fitted = Fit[dataPercentageAliveLog, {1, t}, t]
```

```
Out[28]= 0.0733202 - 1.01457 t
```

```
In[29]:= g2 = Plot[fitted, {t, 0, 4}]
```



```
In[30]:= Show[g1, g2]
```



```
In[31]:= Exp[-1.01457]
```

```
Out[31]= 0.362558
```

Annual survival probability of robins is 0.36.

■ Doubling time

For an exponential population growth, we can define the time necessary for the population size to be doubled. This is called "doubling time". Doubling time T_B can be solved as follows.

$$N_{t+T} = N_0 r^{t+T} = 2 N_t = 2 N_0 r^t$$

From this, the doubling time $T = \log 2 / \log r$

When $r = 1.05$ as above, the double time $T = \log 2 / \log 1.05$

```
In[32]:= Log[2] / Log[1.05]
```

```
Out[32]= 14.2067
```

It takes about 14 years for the exponential growth with common ratio $r = 1.05$ per year (annual 5% increase).

■ Half life

For an exponential population decline we can define the time necessary for the population size to be halved. This is called "half life". Half life T_H can be solved as follows.

$$N_{t+T} = N_0 r^{t+T} = N_t/2 = N_0 r^t/2$$

From this, half life $T = -\log 2 / \log r$

When $r = 0.95$, the half life $T = -\log 2 / \log 0.95$

```
In[33]:= -Log[2] / Log[0.95]
```

```
Out[33]:= 13.5134
```

It takes about 13.5 years for the exponential decline with common ration $r = 0.95$ per year (5% decline annually).

Questions

Annual growth rate of the world population in 2015 is estimated as 1.07 %. If this rate is kept constant, how long will it take for the world population to be doubled, e.g., from 7.3 billion to 14.6 billion?

```
In[34]:= CountryData["World", "Population"]
```

```
Out[34]:= 7.13001 × 109 people
```

```
In[35]:= Log[2] / Log[1.0107]
```

```
Out[35]:= 65.1261
```

Annual growth rate of Bangladesh population in 2015 is estimated as 1.61 %. If this rate is kept constant, how long will it take for Bangladesh population to be doubled, e.g., from 156 million to 312 million?

```
In[36]:= CountryData["Bangladesh", "Population"]
```

```
Out[36]:= 156 380 192 people
```

```
In[37]:= Log[2] / Log[1.0161]
```

```
Out[37]:= 43.3983
```

Annual growth rate of Japanese population in 2015 is estimated as -0.17%. If this rate is kept constant, how long will it take for Japanese population to be halved e.g., from 126 million to 63 million?

```
In[38]:= CountryData["Japan", "Population"]
```

```
Out[38]:= 126 225 259 people
```

```
In[39]:= -Log[2] / Log[0.9983]
```

```
Out[39]:= 407.387
```

Data from the United States Census Bureau :

[http : // www.census.gov/population/international/data/idb/region.php?N = %20 Results %20 & T = 11 & A = both & RT = 0 & Y = 2016 & R = 1 & C =](http://www.census.gov/population/international/data/idb/region.php?N=%20Results%20&T=11&A=both&RT=0&Y=2016&R=1&C=)