Lecture 6: Birth-death models

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1 A deterministic model of birth and death

Consider a population and imagine that each individual gives birth to an offspring or dies according a certain rule. Let the population size be N and we focus on the dynamics of N as a function of time t. If both birth and death occur at a constant rate, say, β and δ, respectively, the net change of the population size N within a short time interval Δt is given as

$$\Delta N = N(t + \Delta t) - N(t) = \beta \Delta t N - \delta \Delta t N$$

because there are N individuals and each individual gives birth and dies with the rate $\beta \Delta t$ and $\delta \Delta t$, respectively.

By arranging this equation and letting $\Delta t \to 0$ we have a differential equation

$$\frac{dN}{dt} = (\beta - \delta)N \tag{1}$$

The solution is

$$N(t) = N(0) \exp[(\beta - \delta)t] \tag{2}$$

where $N(0)$ is initial population size.

It is obvious that the population size N exponentially increases when the birth rate exceeds the death rate, $\beta > \delta$, and it exponentially decreases toward zero when $\beta < \delta$. If we take the logarithm of $N(t)$, it increases or decreases linearly with time t and the slop is given by $\beta - \delta$ because $\log N(t) = \log N(0) + (\beta - \delta)t$.

This is a deterministic model of birth and death. $\beta - \delta$ is the net rate of increase per individual. As in the previous deterministic immigration-emigration model the population size should be interpreted as "density", not the number of individuals as non-negative integer. The difference is that α and β in the immigration-emigration model are now replaced with βN and δN where N is the population size as density. We now want to derive a stochastic model that corresponds to this deterministic model.

2 A stochastic model

We assume that the birth rate β is the probability that an individual gives birth to an offspring and that the death rate δ the probability that the individual dies within a unit time. We also assume that within a short time interval Δt, only one of the following three cases occurs mutually exclusively; an individual 1) gives birth to an offspring, 2) dies, or 3) neither gives birth nor dies. This stochastic birth-death process could be implemented using the algorithm with a constant time interval Δt small enough.

> For all individuals repeat
> 1) Give birth to a new individual with probability $\beta\Delta t$.
> 2) Remove this individual with probability $\delta\Delta t$.

Here is an outline of the program that simulates this stochastic process.

```
#define BIRTH_RATE 0.03
#define DEATH_RATE 0.02
#define DT  0.1
#define INTV 10
main()
{
  int pop_size, new_indiv, dead_indiv, i, step;
  double prob_birth, prob_death, ran;

  prob_birth = BIRTH_RATE* DT;
  prob_death = DEATH_RATE * DT;
  pop_size = 10;   /* initial population size */

  for(step=0; step<5000; step++){    /* advance the time by DT */

     if( step%INTV == 0) printf("%d ", pop_size);
     new_indiv = 0;
     dead_indiv = 0;
     for(i=1; i<=pop_size; i++){  /* for all individuals */
       ran = genrand_real1();
       if( ran < prob_birth )
         new_indiv++;
       else if( prob_birth < ran && ran < prob_birth + prob_death )
         dead_indiv++;

    } /* end of for i */

    pop_size += (new_indiv - dead_indiv) ;
  } /* end of for step */
}
```

3 Waiting time

As in the stochastic immigration-emigration model, it would be better to introduce waiting time to run simulation. Either birth or death takes place with the rate $\beta N + \delta N$, and waiting time to the next event (either birth or death) w is given as an exponential distributed random variable those p.d.f is

$$f(w) = \lambda \exp[-\lambda w]$$

where $\lambda = \beta N + \delta N$. A birth occurs with the conditional probability $\beta N/(\beta N + \delta N) = \beta/(\beta + \delta)$ and a death likewise. For the sake of calculating average population size later, it is convenient to output the population size N with an equal time interval $\Delta T = 1$.

4 Simulation

1. Write a C program to carry out simulation of the stochastic birth-death process. In the simulation we start with an initial population size, say, $n(0) = 10$ and repeat the dynamics with the same initial condition for several times. The dynamics of the population size should be written into a file. The data should be separated by a white space and write them in one line in the following format (assuming the time interval is 1).

Trial 1:	$n(0)$	$n(1)$	$n(2)$	$\cdots$	$n(100)$
Trial 2:	$n(0)$	$n(1)$	$n(2)$	$\cdots$	$n(100)$
Trial 3:	$n(0)$	$n(1)$	$n(2)$	$\cdots$	$n(100)$
$\vdots$					

2. Using *Mathematica*, draw the simulated dynamics both in normal and logarithmic scale to see how the population size changes. Observe that in some trial the population size N can become zero at some time t and and hereafter it remains zero. Consider why the population size N remains zero once it reached zero.