Lecture 5: Immigration-emigration models #4

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1 Probability generating function

The population size n is a discrete random number and is associated with a probability distribution $P_n(t)$ which evolves with time t according to master equation. Solving $P_n(t)$ in general is not always easy but for some simple case we can do it. We now introduce **probability generating function** $G(t,z)$ associated with a probability distribution $P_n(t)$ as

$$G(t,z) = \sum_n P_n(t) z^n \tag{1}$$

where the summation is taken for all possible n. This series will converge when $|z| < r$ where r is convergence radius. Probability generating function is just a power series whose i-th coefficient is $P_i(t)$.

Let's take a look of general properties of probability generating function. It is obvious that $G(t,z)$ evaluated at $z = 1$ is always 1 because it is just summation of probability $P_n(t)$.

$$G(t,1) = \sum_n P_n(t) = 1 \tag{2}$$

Differentiating (2) with z yields

$$\frac{\partial}{\partial z} G(t,z) = \sum_n n P_n(t) z^{n-1}$$

and by substituting $z = 1$ we have a useful result

$$\left. \frac{\partial}{\partial z} G(t,z) \right|_{z=1} = \sum_n n P_n(t) = \langle n \rangle = E[n] \tag{3}$$

Similarly differentiating (2) with z twice

$$\frac{\partial^2}{\partial z^2} G(t,z) = \sum_n n(n-1) P_n(t) z^{n-2}$$

and substituting $z = 1$ gives

$$\frac{\partial^2}{\partial z^2} G(t,z)\bigg|_{z=1} = \sum_n n(n-1)P_n(t) = E[n(n-1)] = \langle n^2 \rangle - \langle n \rangle \tag{4}$$

We now remember that $Var[n] = \langle n^2 \rangle - \langle n \rangle^2$. That is,

$$Var[n] = \left\{ \frac{\partial^2}{\partial z^2} G(t,z) + \frac{\partial}{\partial z} G(t,z) - \left(\frac{\partial}{\partial z} G(t,z) \right)^2 \right\} \bigg|_{z=1} \tag{5}$$

These calculations show that if we can solve and obtain a probability generating function $G(t, z)$, we can derive the ensemble average $\langle n \rangle$ and the variance $Var[n]$ from $G(t, z)$. In previous lectures, we derived moment dynamics directly from master equation. But they can be also obtained from $G(t, z)$. Even more, probability $P_n(t)$ is given as a coefficient of Taylor expansion of $G(t, z)$ around $z = 0$. This means solving $G(t, z)$ is equivalent to solving $P_n(t)$. In the following sections we try to solve the p.g.f. $G(z, t)$ of the stochastic immigration-emigration process.

2 Solving the pgf of immigration-emigration process

We now solve the pgf of immigration-emigration process in which n can be negative (population size is no longer restricted non-negative). The master equation is

$$\frac{dP_n(t)}{dt} = \alpha P_{n-1}(t) + \beta P_{n+1}(t) - (\alpha + \beta)P_n(t) \quad \text{for } -\infty < n < \infty \tag{6}$$

and the pgf is defined as

$$G(t,z) = \sum_n P_n(t) z^n \tag{7}$$

Differentiating the pgf with t yields

$$\frac{\partial}{\partial t} G(t,z) = \sum_n \frac{d}{dt} P_n(t) z^n$$

Using the master equation (6), we have

$$\begin{aligned} \frac{\partial}{\partial t} G(t,z) &= \sum_n \left\{ \alpha P_{n-1}(t) + \beta P_{n+1}(t) - (\alpha + \beta) P_n(t) \right\} z^n \\ &= \alpha z \sum_n P_{n-1}(t) z^{n-1} + \frac{\beta}{z} \sum_n P_{n+1}(t) z^{n+1} - (\alpha + \beta) \sum_n P_n(t) z^n \\ &= (\alpha z + \beta/z - \alpha - \beta) G(t,z) \end{aligned}$$

This is ODE of $G(t, z)$ with respect to time t and has unique solution with initial condition $G(0, z) = z^m$ where m is initial population size at $t = 0$, $n(0)$. This can be readily solved by variable separation. The solution is

$$G(t,z) = z^m \exp\left[(-\alpha - \beta + \alpha z + \beta/z)t\right] \tag{8}$$

3 Problem

We have solved the pgf of the immigration-emigration process $G(t, z)$.

1. Confirm that the expected value $E[n]$ and variance $Var[n]$ of n derived from the moment dynamics in the last lecture coincide with those derived from the pgf.
2. By Taylor expanding the pgf $G(t, z)$ with respect to z and looking at coefficients of z^n, $P_n(t)$ can be obtained. This is not actually easy but we are a bit close to the solution $P_n(t)$.