Lecture 4: Immigration-emigration models #3

Fugo Takasu
Dept. Information and Computer Sciences
Nara Women's University
takasu@ics.nara-wu.ac.jp

23 May 2011

1 Moment dynamics

We have derived the master equation for the stochastic immigration-emigration process. $P_n(t)$ is the probability that the population size is n at time t. It obeys the following master equation.

$$\frac{dP_n(t)}{dt} = \alpha P_{n-1}(t) + \beta P_{n+1}(t) - (\alpha + \beta) P_n(t) \quad \text{for } n \geq 1 \tag{1}$$

$$\frac{dP_0(t)}{dt} = \beta P_1(t) - \alpha P_0(t) \tag{2}$$

Instead of solving the master equations (1) and (2) with a certain initial condition, we now focus on the dynamics of expected value and variance of n, which turns out to be relatively easy task if we allow the population size n can be negative. Negative n makes no biological sense, but removing the constraint that n should be non-negative merits as we shall see. Now equation (1) is valid for all $n = 0, \pm 1, \pm 2, \cdots$. Before we do this, we introduce a useful notation.

The quantity

$$\langle n^k \rangle = \sum_n n^k P_n$$

where $k = 0, 1, 2, 3, \cdots$ is called the k-th moment of a random variable n where P_n is the probability distribution of n. The summation is taken for all possible n.

The zero-th moment is always 1 because it is just the sum of the probability P_n. The first moment is the expected value of n, $E[n]$, (mean or average) , and the second moment is the variance of n, $Var[n]$, plus squared mean.

$$\langle n^0 \rangle = 1$$
$$\langle n^1 \rangle = E[n]$$
$$\langle n^2 \rangle = Var[n] + \langle n \rangle^2$$

2 First moment dynamics

Let us first derive the dynamics of the first moment $\langle n \rangle = \sum nP_n$. We multiply equation (1) with n and summing up for all possible n. The right hand side is

$$\sum_n n\frac{dP_n(t)}{dt} = \frac{d}{dt}\sum_n nP_n(t) = \frac{d\langle n\rangle}{dt}$$

Here we exchanged the order of taking summation and differentiation. This exchange is not always possible but we can do this for well-behaving functions. The left hand side is

$$\begin{aligned}
&\sum_n \{\alpha nP_{n-1}(t) + \beta nP_{n+1}(t) - (\alpha+\beta)nP_n(t)\} \\
&= \alpha\sum_n \{nP_{n-1}(t) - nP_n(t)\} + \beta\sum_n \{nP_{n+1}(t) - nP_n(t)\} \\
&= \alpha\sum_n \{(n+1)P_n(t) - nP_n(t)\} + \beta\sum_n \{(n-1)P_n(t) - nP_n(t)\} \\
&= \alpha\sum_n P_n(t) - \beta\sum_n P_n(t) \\
&= \alpha - \beta
\end{aligned}$$

Here we used identity $\sum_n nP_{n-1} = \sum_n (n+1)P_n$ where summation is for all possible n. Then we have the dynamics of the first moment, expected value of the population size $\langle n\rangle$ as

$$\frac{d\langle n\rangle}{dt} = \alpha - \beta \tag{3}$$

This is exactly the same differential equation for the deterministic model of immigration and emigration. This means that the expected value of n just follows the same dynamics of the corresponding deterministic process. If we run simulation many times and calculate the average for each time t, the average (we call this ensemble average) should linearly increase or decrease as time passes.

3 Second moment dynamics

In the same way we derive the second moment dynamics by multiplying equation (1) with n^2 and summing up for all possible n.

The left hand side is

$$\sum_n n^2\frac{dP_n(t)}{dt} = \frac{d}{dt}\sum_n n^2P_n(t) = \frac{d\langle n^2\rangle}{dt}$$

and the right hand side is

$$
\begin{aligned}
&\sum_n \left\{\alpha n^2 P_{n-1}(t) + \beta n^2 P_{n+1}(t) - (\alpha+\beta) n^2 P_n(t)\right\} \\
&= \alpha \sum_n \left\{n^2 P_{n-1}(t) - n^2 P_n(t)\right\} + \beta \sum_n \left\{n^2 P_{n+1}(t) - n^2 P_n(t)\right\} \\
&= \alpha \sum_n \{(n+1)^2 P_n(t) - n^2 P_n(t)\} + \beta \sum_n \{(n-1)^2 P_n(t) - n^2 P_n(t)\} \\
&= \alpha \sum_n (2n+1) P_n(t) + \beta \sum_n (-2n+1) P_n(t) \\
&= 2\alpha \sum_n n P_n(t) - 2\beta \sum_n n P_n(t) + \alpha + \beta \\
&= 2(\alpha-\beta)\langle n\rangle + \alpha + \beta
\end{aligned}
$$

That is,

$$\frac{d\langle n^2\rangle}{dt} = 2(\alpha-\beta)\langle n\rangle + \alpha + \beta \tag{4}$$

$Var[n] = \langle n^2\rangle - \langle n\rangle^2$, so the dynamics of the variance is given as

$$
\begin{aligned}
\frac{d}{dt}Var[n] &= \frac{d}{dt}\langle n^2\rangle - \frac{d}{dt}\langle n\rangle^2 \\
&= \frac{d}{dt}\langle n^2\rangle - 2\langle n\rangle \frac{d}{dt}\langle n\rangle \\
&= 2(\alpha-\beta)\langle n\rangle + \alpha + \beta - 2(\alpha-\beta)\langle n\rangle \\
&= \alpha + \beta
\end{aligned}
$$

The variance linearly increases with time because $\alpha + \beta > 0$. We have seen in simulation where initial state is set the same that the variance continues to increase as this analytical result suggests.

We now have derived the dynamics of $E[n]$ and $Var[n]$ for the population size n that is a stochastic variable.

$$\frac{d}{dt}E[n] = \alpha - \beta \tag{5}$$

$$\frac{d}{dt}Var[n] = \alpha + \beta \tag{6}$$

4 Problem

1. Run the simulation and record the time sequence of the population size n for 100 time step for many times. To make easy the calculation of average and variance, only population size should be written into the file separated with a white space.

   Draw the dynamics of the average and the variance of the population size using *Mathematica*. Compare the simulation results and the analytic predictions of the average and the variance of the population size (5) and (6). Here in the simulation we allow “negative” population size to have good match.